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ASYMPTOTIC SOLUTION OF SPATIAL PROBLEMS OF ELASTICITY THEORY ABOUT EXTENDED PLANE SHEAR CRACKS*

L.B. KOREL'SHTEIN

An asymptotic solution is obtained for spatial problems of elasticity theory concerning shear cracks occupying a plane domain extended along a certain curve. Terms of the expansion of the solution in a small parameter characterizing the extension of the crack are constructed on the basis of a system of integrodifferential Eqs.(1) in the displacement component of points of the crack surfaces. On the basis of the asymptotic formulas obtained, a dependence of the displacement of the surfaces, the stress intensity factors, and the specific increment of the total potential energy on the crack shape and the load is described, and relationships of the type of congruence theorems are set up /2/. Values of the stress intensity factors and the displacements of the surfaces that are computed for specific kinds of cracks by means of the asymptotic formulas are in agreement with the known exact values of with values obtained by a numerical method.

The asymptotic solution for an analogous simpler problem about separation cracks that reduces to the solution of one integrodifferential equation, is presented in /3/.

1. A crack extended along a line. We consider a homogeneous isotropic medium with a crack occupying the domain G in the plane $x_3 = 0$. Oppositely directed forces

$$\sigma_{i3}^+(x_1, x_2, 0) = \sigma_{i3}^-(x_1, x_2, 0) = -t_i(x_1, x_2), \quad i = 1, 2$$
$$\sigma_{33}(x_1, x_2, 0) = 0, \quad (x_1, x_2) \in G$$

are applied to the crack surfaces (the plus and minus superscripts correspond to the upper and lower edges of the crack). There is no load at infinity. Then the normal components of the displacement of the crack edges are continuous

$$u_3^+(x_1, x_2, 0) = u_3^-(x_1, x_2, 0), \quad (x_1, x_2) \in G$$

and we have for the tangential displacement components

$$u_i^+(x_1, x_2, 0) = -u_i^-(x_1, x_2, 0) = u_i(x_1, x_2), \quad i = 1, 2$$
$$u_i(x_1, x_2) = 0, \quad (x_1, x_2) \overline{\in} G$$

The determination of shear crack surface displacements reduces to seeking the bounded function $\mathbf{u}(x_1, x_2) = u_1(x_1, x_2)\mathbf{e}_1 + u_2(x_1, x_2)\mathbf{e}_2$ which equals zero outside the domain G and satisfies the integrodifferential equation /1/

$$P_G\{F_{x_1x_2}^{-1}[|\xi| A * \mathbf{u}(x_1, x_2)]\} = \beta \mathbf{t}(x_1, x_2), \quad \beta = (1-\nu)/\mu \tag{1.1}$$

Here A is a second-order matrix with the elements a_{ij}, $a_{11} = 1 - \nu\eta_2^2$, $a_{22} = 1 - \nu\eta_1^2$, $a_{12} = a_{21} = \nu\eta_1\eta_2$, $\eta_i = \xi_i/|\xi|$, $F_{x_1x_2}$ is the two-dimensional Fourier transform

$$F_{x_1x_2}[\varphi(x_1, x_2)] = \iint_{-\infty}^{\infty} \exp[i(\xi, \mathbf{x})]\varphi(x_1, x_2)\,dx_1\,dx_2$$

P_G is the contraction operator in the domain G; the functions u_i, a_{ij}, t_i are understood to be generalized from the spaces $S'(R^2)$ and $S'(G)$ respectively, μ and ν are the shear modulus and Poisson's ratio of the medium, and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the directions of the axes of the $Ox_1x_2x_3$ coordinate system.

Eq.(1.1) can be written in x-space in the form

$$(1-\nu)\Delta\Psi + \nu\nabla(\nabla\cdot\Psi) = 2\pi\beta\mathbf{t}, \quad \mathbf{x} \in G \tag{1.2}$$

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$$\Psi(x_1, x_2) = \iint\limits_G \frac{\mathbf{u}(x_1', x_2')}{r} dx_1' dx_2', \qquad r^2 = (x_1' - x_1)^2 + (x_2' - x_2)^2$$

We consider a crack occupying the domain $G(\varepsilon)$ of the following kind (Fig.1) $|x_1| \leqslant L$, $|x_2| \leqslant \varepsilon\rho(x_1)$ where $L > 0$, the function $\rho(x_1)$ is bounded and $\rho(x_1) \in C^3(-L, L)$, $\rho > 0$ while $\varepsilon > 0$ is a dimensionless parameter. For small ε we obtain a crack in the shape of a narrow strip extended along the Ox_1 axis. The problem is to determine the asymptotic form of the crack surface displacement $\mathbf{u}(x_1, x_2, \varepsilon)$ (corresponding to the crack $G(\varepsilon)$) as $\varepsilon \to 0$.

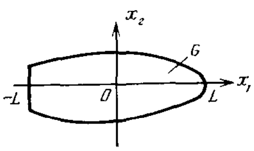

Fig.1

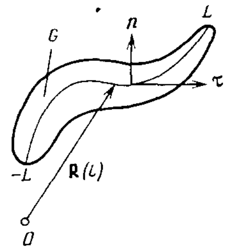

Fig.2

We introduce the internal coordinate $Y = x_2/\varepsilon$. Proceeding as in /3/, we write (1.1) in x_1, Y coordinates in the form

$$\begin{gathered}P_G\{[\Phi_0 + \varepsilon\Phi_1 + \varepsilon^2(\ln 2/\varepsilon)\Phi_2^* + \varepsilon^2\Phi_2 + o(\varepsilon^2)] * \mathbf{u}(x_1, \\ Y, \varepsilon)\} = -2\pi\beta\varepsilon\mathbf{t}(x_1, Y, \varepsilon)\end{gathered} \tag{1.3}$$

$$\Phi_0 = -2\delta(x_1)\frac{\partial}{\partial Y} P\frac{1}{Y}\operatorname{diag}(1-\nu, 1)$$

$$\Phi_2^* = \delta''(x_1)\operatorname{diag}(1+\nu, 1-2\nu)$$

$$\Phi_2 = \frac{1}{2}\delta''(x_1)\operatorname{diag}(1-\nu, 1) - \left[\delta''(x_1)\ln|Y| - \frac{1}{2}\frac{\partial^2}{\partial x_1^2} P\frac{1}{|x_1|}\right]\operatorname{diag}(1+\nu, 1-2\nu)$$

$$\Phi_1 = -2\nu\delta'(x_1) P\frac{1}{Y} C$$

where C is a second-order square matrix with elements $c_{ij} = 1 - \delta_{ij}$

$$\left(P\frac{1}{|x_1|}, \varphi\right) = \iint\limits_{-\infty}^{\infty} [\varphi(x_1, Y) - \theta(1 - |x_1|)\varphi(0, Y)]\frac{dx_1 dY}{|x_1|}, \quad \varphi \in S(R^2)$$

We will examine the problem of crack perturbation of an external stress field $\sigma_{ij}(x_1, x_2, x_3)$ (which takes place in an infinite medium without cracks)

$$\mathbf{t}(x_1, Y, \varepsilon) = \mathbf{t}(x_1, x_2) = \mathbf{t}(x_1, \varepsilon Y) = \sum_{i=0}^{2} \frac{\varepsilon^i \mathbf{t}^i(x_1) Y^i}{i!} + o(\varepsilon^2)$$

$$\mathbf{t}^i(x_1) = [\partial^i \mathbf{t}(x_1, x_2)/\partial x_2^i]_{x_2=0}, \quad t_i(x_1, x_2) = \sigma_{i3}(x_1, x_2, 0)$$

It is natural to seek the asymptotic form $\mathbf{u}(x_1, Y, \varepsilon)$ in the following form that results from comparing asymptotic expansions of the kernel and the right-hand side of (1.3):

$$\begin{gathered}\mathbf{u}(x_1, Y, \varepsilon) = \sum_{i=0}^{2} \varepsilon^{i+1}\mathbf{u}^i(x_1, Y) + \varepsilon^3 \ln\frac{2}{\varepsilon}\mathbf{v}(x_1, Y) + \\ \varepsilon\mathbf{w}(x_1, Y, \varepsilon) + o(\varepsilon^3)\end{gathered} \tag{1.4}$$

Here the conditions on the functions u^i, v are the same as in /3/. Proceeding as in /3/, we obtain at the middle section of the crack

$$\Phi_0 * \mathbf{u}^i = -2\pi\beta\frac{\mathbf{t}^i Y^i}{i!} - \sum_{k=1}^{i}\Phi_k * \mathbf{u}^{i-k}, \quad \Phi_0 * \mathbf{v} = -\Phi_2^* * \mathbf{u}^0, \quad i = 0, 1, 2$$

The equations obtained are the equations of a plane crack of longitudinal and transverse shear and can be solved sequentially in quadratures. We consequently obtain

$$
\begin{aligned}
&u_i(x_1, Y, \varepsilon) = \varepsilon\mu^{-1}\lambda_i \sqrt{\rho^2(x_1) - Y^2}\, Q_i(x_1, Y, \varepsilon) + o(\varepsilon^3) \\
&Q_i(x_1, Y, \varepsilon) = t_i^0 + 0{,}5\varepsilon Y Q_{i1} + \varepsilon^2 (2Y^2 + 1) \times \\
&\quad (Q_{i2} - 0{,}5\alpha_i^1 \lambda_i^{-1} t^{0''})/12 + 0{,}125\varepsilon^2 Q_i^* \\
&Q_{i(n+1)} = t_i^{n+1} - \nu\lambda_i^{-1} t_j^{n'} \\
&Q_i^* = \alpha_i^2 \lambda_j^{-1} [\Lambda_i + T(q_{0i}^{00})]'' + \\
&\quad 2\nu [\lambda_i^{-1} q_{1j}^{01} - \nu q_{1j}^{11}/(1-\nu)] + 2\lambda_i \lambda_j^{-1} q_{0j}^{02} \\
&f = \rho^2, \quad g(x_1) = L^2 - x_1^2, \quad \lambda = (1, 1-\nu) \\
&\alpha^1 = (1 + 2\nu, 1 - 3\nu), \quad \alpha^2 = (1 + \nu, 1 - 2\nu) \\
&q_{mn}^{kl} = \left[\frac{d^k t_n^m}{dx_1^k} f\right]^{(l)}, \quad \Lambda = \varphi f \ln \frac{16g}{\varepsilon^2 f}, \quad T\varphi = \int_{-L}^{L} \frac{\varphi(x_1') - \varphi(x_1)}{|x_1' - x_1|}\, dx_1'
\end{aligned}
\tag{1.5}
$$

Here and henceforth, unless otherwise specified, the pair (i, j) take the values (1.2), (2.1).

From (1.5) we obtain the asymptotic form

$$
\begin{aligned}
&K_{i+1}(x_1, \pm\varepsilon\rho(x_1)) = \pm K_0(x_1) [Q_j(x_1, \pm\rho(x_1), \varepsilon) \pm \\
&\quad (-1)^i \lambda_j \lambda_j^{-1} \rho'(x_1) Q_i(x_1, \pm\rho(x_1), \varepsilon) + o(\varepsilon^2)] \\
&K_0(x_1) = \sqrt{\pi\varepsilon\rho(x_1)}\, [1 + \varepsilon^2 \rho'^2(x_1)]^{-1/4}
\end{aligned}
\tag{1.6}
$$

for the stress intensity factors K_2 and K_3 of transverse and longitudinal shear at the points $(x_1, \pm\varepsilon\rho(x_1))$ of the crack contour (for $x_1 \in (-L, L)$.

2. A crack extended along a curve. We will now examine the more general case of a crack extended along a certain smooth curve, given in the plane $x_3 = 0$, without selfintersections $\mathbf{R}(l)$ and of length $2L$, where l is the distance along the curve from its middle point along the length, $l \in [-L, L]$ (closed curves with $\mathbf{R}(-L) = \mathbf{R}(L)$ are also considered allowable). For the directions $\boldsymbol{\tau}(l), \mathbf{n}(l)$ tangential and normal to the curve, we have

$$
\begin{aligned}
&d\mathbf{R}(l)/dl = \boldsymbol{\tau}(l), \quad \mathbf{n}(l) = \mathbf{e}_3 \times \boldsymbol{\tau}(l) \\
&d\boldsymbol{\tau}(l)/dl = -k(l)\,\mathbf{n}(l), \quad d\mathbf{n}(l)/dl = k(l)\,\boldsymbol{\tau}(l)
\end{aligned}
\tag{2.1}
$$

where $k(l)$ is the curvature of the curve at the point $\mathbf{R}(l)$ (positive of negative). We introduce an orthogonal system of coordinates

$$
\mathbf{x}(l, m) = \mathbf{R}(l) + \varepsilon m \mathbf{n}(l)
\tag{2.2}
$$

in the $x_3 = 0$ plane.

We shall examine cracks whose domain $G(\varepsilon)$ (Fig.2) is given by the inequalities $|l| \leqslant L$, $|m| \leqslant \rho(l)$ (the conditions on $\rho(l)$ are the same as in Sect.1). As in Sect.1, the problem is to determine the asymptotic form of the surface displacement $\mathbf{u}(l, m, \varepsilon)$ and $\varepsilon \to 0$.

If we introduce the coordinate system (l, M), $M = m/\rho(l)$, then the mapping Jacobian $(l, M) \to (x_1, x_2)$ equals

$$
D(l, M, \varepsilon) = \varepsilon\rho(l)\,[1 + \varepsilon M \rho(l)\, k(l)]
\tag{2.3}
$$

and (1.2) is written in the domain $G(\varepsilon)$ in the form

$$
\Omega(l, m, k(l), \boldsymbol{\tau}(l), \mathbf{n}(l), \varepsilon, \partial/\partial l, \partial/\partial m)\, \boldsymbol{\Psi} = -2\pi\beta \mathbf{t}(x), \ \mathbf{x} \in G(\varepsilon)
\tag{2.4}
$$

$$
\begin{aligned}
&\boldsymbol{\Psi}(l, M, \varepsilon) = K\mathbf{u} = \int_{-L}^{L}\int_{-1}^{1} \frac{\mathbf{u}(l', M', \varepsilon)}{|\Delta\mathbf{x}|} D(l', M', \varepsilon)\, dl'\, dM' \\
&\Delta\mathbf{x} = \mathbf{x}(l', M') - \mathbf{x}(l, M)
\end{aligned}
\tag{2.5}
$$

where Ω is a certain differential operator with variable coefficients (whose explicit expression is not given here because of its awkwardness). Because of (2.3) we have for the operator K defined by (2.5)

$$
K\varphi = \varepsilon H(\varphi\rho) + \varepsilon^2 H(\varphi M \rho^2 k), \quad H\varphi = \int_{-L}^{L}\int_{-1}^{1} \frac{\varphi(l', M')}{|\Delta\mathbf{x}|}\, dl'\, dM'
\tag{2.6}
$$

Therefore, the determination of the asymptotic form $\boldsymbol{\Psi}(l, m, \varepsilon)$ as $\varepsilon \to 0$ reduces to determining the asymptotic form of the operator H presented in /3/ (there is a misprint in

(2.11) in /3/ in the last integral in the braces; the term $m'^2\rho'^2/2$) is omitted).

Let $\mathbf{t}(l, m, \varepsilon)$ have the power-law asymptotic form

$$\mathbf{t}(l, m, \varepsilon) = \sum_{i=0}^{2} \varepsilon^i \boldsymbol{\tau}^i(l, m) + o(\varepsilon^2), \quad \boldsymbol{\tau}^i \in C^3(\bar{G}) \tag{2.7}$$

We will seek the asymptotic form $\mathbf{u}(l, m, \varepsilon)$ in the middle section of the crack in the form of (1.4) (by replacing the coordinates x_1, Y by l, m and omitting terms with $\mathbf{w}$ governing the boundary layer at the ends). Utilizing (2.6) and (2.7), the asymptotic form H and proceeding as in /3/, we can obtain equations for $\mathbf{u}^i$ and $\mathbf{v}$ which reduce, after simplification, to the form

$$\begin{aligned}
&\lambda_j P(u_i^0)''_{mm} = \pi\beta\tau_i^0 \\
&2\lambda_j P(u_i^1)''_{mm} = 2\pi\beta\tau_i^1 - \lambda_j k P(u_i^0)'_m - 2\nu P(u_j^0)''_{ml} \\
&2\lambda_j P(v_i \ln 2/\varepsilon + u_i^2)''_{mm} = 2\pi\beta\tau_i^2 - \lambda_j k P(u_i^1)'_m - \\
&\quad 2\nu P(u_i^1)''_{ml} + 0{,}5\alpha_i^2 P_\rho(u_i^0)''_{ll} - (1)^i\, 0{,}5\lambda_i k' P_\rho(u_j^0) - \\
&\quad (-1)^i (1 - 0{,}5\nu) k P_\rho(u_j^0)'_l - 0{,}375\lambda_j k^2 P_\rho(u_i^0) - \alpha_i^1 (U_i^0)''_{ll} + \\
&\quad (-1)^i (2 - \nu + \lambda_i) k' U_j^0/3 + (-1)^i (1 + \lambda_j) k (U_j^0)'_l + \\
&\quad (5 - \nu - 3\delta_{1i}) k^2 U_i^0/12 + \lambda_j k^2 m P(u_i^0)'_m + \\
&\quad 2\nu\delta_{2i} k m P(u_j^0)''_{ml} + (1 - 2\nu) I_i(\mathbf{U}^0) + 3\nu J_i(\mathbf{U}^0) \\
&\mathbf{U}^0(l) = \int_{-\rho(l)}^{\rho(l)} \mathbf{u}^0(l, m)\, dm, \quad P(\varphi) = \int_{-\rho(l)}^{\rho(l)} \varphi(l, m') \ln|\Delta m|\, dm' \\
&P_\rho(\varphi) = \int_{-\rho(l)}^{\rho(l)} \varphi(l, m') \ln \frac{4g}{\varepsilon^2 (\Delta m)^2}\, dm', \quad g = L^2 - l^2 \\
&\mathbf{I}(\mathbf{U}^0) = \int_{-L}^{L} \{\mathbf{U}^0(l') |\Delta\mathbf{R}|^{-3} - |\Delta l|^{-3} Z_2[|\Delta l|^3 |\Delta\mathbf{R}|^{-3} \mathbf{U}^0(l')]_{l'=l}\}\, dl' \\
&\mathbf{J}(\mathbf{U}^0) = \int_{-L}^{L} \{(\Delta\mathbf{R}, \mathbf{U}^0(l')) |\Delta\mathbf{R}|^{-5} \Delta\mathbf{R} - |\Delta l|^{-3} \times Z_2[|\Delta l|^3 |\Delta\mathbf{R}|^{-5} (\Delta\mathbf{R}, \mathbf{U}^0(l')) \Delta\mathbf{R}]_{l'=l}\}\, dl'
\end{aligned} \tag{2.8}$$

where $Z_2[\varphi(l')]_{l'=l}$ is a second-order Taylor polynomial of the function $\varphi(l')$ at the point $l' = l$, and the subscripts 1 and 2 of $\mathbf{u}^k$, $\mathbf{v}$, $\mathbf{I}$, $\mathbf{J}$, $\mathbf{U}^0$ denote components of the corresponding vectors in the directions $\boldsymbol{\tau}(l)$ and $\mathbf{n}(l)$ (such notation is later used everywhere where cracks along the curve are analysed). For each fixed l Eqs.(2.8) (like their special case (1.4)) are the equations of a plane shear crack and can be solved successively in quadratures.

Let $\mathbf{t}(l, m, \varepsilon) = \mathbf{t}(\mathbf{x})$ (the problem of external stress field perturbation by a crack). Then $\boldsymbol{\tau}^i(l, m) = \mathbf{t}^i(l) m^i/i!$, $\mathbf{t}^i(l) = [d^i\mathbf{t}/d\mathbf{n}(l)^i]_{\mathbf{x}=\mathbf{R}(l)}$ and we obtain from (2.8)

$$\begin{aligned}
&u_i = \varepsilon\mu^{-1}\lambda_i \sqrt{\rho^2(l) - m^2}\, Q_i(l, m, \varepsilon) \\
&Q_i(l, m, \varepsilon) = t_i^0 + 0{,}25\varepsilon m Q_{i1} + \varepsilon^2 m^2 \{Q_{i2} - \alpha_i^1 \lambda_i^{-1} t_i^{0''} + 3{,}25 k^2 t_i^0 + \\
&\quad (-1)^i \lambda_i^{-1} [(4 - \alpha_i^1 - \alpha_2^2) k t_j^{0'} - \lambda_j k' t_j^0]\}/12 + \\
&\quad \varepsilon^2 [Q_i^+/48 + (-1)^i Q_i^\pm/24 + 0{,}25(1 - 2\nu) I_i(f\gamma) + 0{,}75\nu J_i(f\gamma)] \\
&\boldsymbol{\gamma}(l) = t_1^0 \boldsymbol{\tau}(l) + (1 - \nu) t_2^0 \mathbf{n}(l), \quad Q_{i(n+1)} = 2t_i^{n+1} - k t_i^n - 2\nu\lambda_i^{-1} t_j^{n'} \\
&Q_i^+ = 6\alpha_i^2 \lambda_j^{-1} \Lambda_i'' - 4{,}5 k^2 \Lambda_i + 4(q_{2i}^{01} + k q_{1i}^{00}) + \\
&\quad \lambda_i^{-1}\nu (12 q_{1j}^{01} + 8 q_{1j}^{10}) - 2\alpha_i^1 \lambda_i^{-1} q_{0i}^{20} - 6(\alpha_i^1 + \alpha_i^2 - \lambda_i) \lambda_j^{-1} q_{0i}^{02} - \\
&\quad 12\nu^2 (1 - \nu)^{-1} q_{0i}^{11} + (3 + \lambda_i \lambda_j^{-1}) k^2 q_{0i}^{00} \\
&Q_i^\pm = -3(2 - \nu) \lambda_i^{-1} k \Lambda_j' - 3k' \Lambda_j + 4k' q_{0j}^{00} + 6\lambda_i^{-1} k q_{0j}^{01} + \\
&\quad 2(\lambda_j + \nu) \lambda_i^{-1} k q_{0j}^{10}
\end{aligned} \tag{2.9}$$

where the derivatives are taken of the vector components.

We obtain the asymptotic

$$\begin{aligned}
&K_{i+1}(l, \pm\rho(l), \varepsilon) = K_0^\pm [(1 \pm \varepsilon\chi) Q_j(l, \pm\rho(l), \varepsilon) \pm \\
&\quad (-1)^i \varepsilon\rho'(l) \lambda_i \lambda_j^{-1} Q_i(l, \pm\rho(l), \varepsilon) + o(\varepsilon^2)] \\
&K_0^\pm = \pm \sqrt{\pi\varepsilon\rho}\, (1 \pm \varepsilon\chi)^{-1/2} [(1 \pm \varepsilon\chi)^2 + \varepsilon^2 \rho'^2]^{-1/4}, \quad \chi(l) = k(l)\rho(l)
\end{aligned} \tag{2.10}$$

for the stress intensity coefficients K_2 and K_3 at the points $(l, \pm\rho(l))$ of the crack contour ($l \in (-L, L)$) from (2.9).

The asymptotic solution obtained possesses the following "locality" property: the first three terms of the asymptotic form $\mathbf{u}$ (to a term on the order of $\varepsilon^3 \ln \varepsilon$) and K_2, K_3 at the crack section $l = l_0$ depend, as is seen from (1.4)-(1.6) and (2.8)-(2.10), only on the local crack parameters $\rho(l_0)$, $\rho'(l_0)$, $\rho''(l_0)$, $k(l_0)$, $k'(l_0)$ and on the given load and its derivatives in the section $l = l_0$ as well as on Poisson's ratio ν. The dependence of the asymptotic form of the solution at a certain section of the crack on the shape of the whole crack domain and the load in its other sections is included in the integrals $\mathbf{I}(f\gamma)$ and $\mathbf{J}(f\gamma)$ or in the expression for $d^2T(t_i^0 f)/dx_1^2$ which can also be expressed in terms of the integral $\mathbf{I}$:

$$d^2T(t_i^0 f)/dx_1^2 = 2I(t_i^0 f) - 3(t_i^0 f)''$$

It is interesting to note that the term with $d^2T(t_i^0 f)/dx_1^2$ vanishes for $\nu = 0.5$ (an incompressible medium) in the asymptotic expression for u_2 in the case of a crack extended along a line (see (1.5)), and the asymptotic form becomes totally "local" and contains no logarithmic term.

3. Examples. 1°. We consider a uniformly loaded crack ($\mathbf{t}(x_1, x_2) = \mathbf{t} = \text{const}$) extended along a line. In this case the Q_i in (1.5) will have the form (we take into account that $t^1 = t^2 = 0$)

$$Q_i(x_1, Y, \varepsilon) = t_i^0 \{1 + 0{,}125\varepsilon^2\lambda_j^{-1}[\alpha_i^2(\Lambda_0 + T(f))'' + 2\lambda_i f'']\}; \quad \Lambda_0 = f \ln(16g\varepsilon^{-2}f^{-1}) \tag{3.1}$$

and according to (1.6) the intensity factors are expressed by the formulas

$$K_{i+1}(x_1 \pm \varepsilon\rho(x_1)) = \pm\sqrt{\varepsilon\rho(x_1)/2}\,[Q_j(x_1, \pm\rho(x_1), \varepsilon) \pm (-1)^i \varepsilon\lambda_i\lambda_j^{-1}\rho'(x_1)t_1 - 0.25\varepsilon^2\rho'^2(x_1)t_j + o(\varepsilon^2)] \tag{3.2}$$

The asymptotic form of the quantities K_2, K_3 at the sites of their greatest values on $(-L, L)$, the points of maximum crack width (where the first term of the asymptotic form (3.2) is a maximum, i.e., $\rho(x_1)$ is a maximum and $\rho'(x_1) = 0$), has the form

$$K_2 = \pm\sqrt{\pi\varepsilon\rho}\,Q_2 + o(\varepsilon^{2,5}), \quad K_3 = \pm\sqrt{\pi\varepsilon\rho}\,Q_1 + o(\varepsilon^{2,5})$$

As is seen from (3.1), the longitudinal and transverse shear problems are separated in the case under consideration of a crack extended along a line uniformly loaded at its middle section, i.e., the i-th component of the surface displacement depends (to $o(\varepsilon^3)$ accuracy) only on the corresponding i-th load component and the crack geometry. Moreover, terms of order ε^2 vanish in the asymptotic form of $\mathbf{u}$ and the factors Q_i are independent of Y. For $\nu = 0.5$ the expression for Q_2 takes an especially simple form $Q_2(x_1, \varepsilon) = t_2^0 \cdot (1 + 0{,}125\varepsilon^2 f'')$. It is also possible to represent (3.1) in the form

$$Q_i(x_1, \varepsilon) = t_i\{1 + \alpha_i^2\lambda_j^{-1}(Q - 1) + 0{,}25\varepsilon^2(\lambda_i - \alpha_i^2)\lambda_j^{-1}f''\} \tag{3.3}$$
$$Q = 1 + 0.125\varepsilon^2\{[\Lambda_0 + T(f)]'' + 2f''\} = 1 + 0.125\varepsilon^2[\Lambda_0'' - f'' + 2I(f)]$$

where Q is analogous to Q_1, Q_2 and is a dimensionless factor in the asymptotic form of the surface displacement of a separation crack of the same shape, loaded by a homogeneous unit load (see /3/, where formulas are presented for Q for the cases of cracks in the shape of an ellipse, a generalized ellipse, and a crack bounded by arcs of parabolas). Note that $Q_i = t_i Q$ for $\nu = 0$.

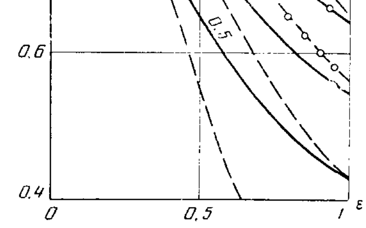

Fig.3

Formulas (1.5), (3.1)-(3.3) provide the possibility of qualitatively representing the behaviour of the surface displacements and the stress intensity factors at the middle part of the crack when its extensibility grows. It can be shown that the displacement components u_1 and u_2 (just like the coefficients K_2 and K_3 at points of maximum crack width) tend to the corresponding quantities of the plane problem (the first terms of the asymptotic form), remaining less in absolute value for small ε than in the convex parts of the crack (where $f'' < 0$) and larger in the concave ($f'' > 0$) parts. Here u_1 and K_3 tend more slowly to quantities of the plane problem than the corresponding quantities for (cleavage) separation cracks, and more slowly the greater the value of ν. The quantities u_2 and K_2 behave oppositely.

We will illustrate the properties noted in examples of cracks of certain specific shapes.

For an elliptical crack ($\rho(x_1) = L\sqrt{1 - x_1^2/L^2}$)

$$Q_i(x_1, \varepsilon) = t_i\{1 - 0.25\varepsilon^2\lambda_j^{-1}[\alpha_i^2 \ln(16\varepsilon^{-2}) - \alpha_i^1 - \alpha_i^2 + \lambda_i]\} \tag{3.4}$$

which yields an asymptotic form of $\mathbf{u}$ that agrees in all its terms with the asymptotic form of the exact solution /4/

$$u_i(x_1, Y) = \varepsilon e_1^2\lambda_i\sqrt{\rho^2(x_1) - Y^2}\,[(1 - \lambda_i\lambda_j^{-1}\varepsilon^2)E(e_1) - (-1)^i\nu\lambda_j^{-1}\varepsilon^2 K(e_1)]^{-1}t_i, \quad e_1^2 = 1 - \varepsilon^2 \tag{3.5}$$

where K, E are the complete elliptic integrals of the first and second kinds.

In Fig.3 we show graphs of the dimensionless transverse shear stress intensity coefficients K_2/K_2° (the four upper curves) and longitudinal shear K_3/K_3° (the four lower curves) on the axis of symmetry of an elliptical crack ($x_1 = 0$) for different values of ν. The solid curves correspond to the exact formulas, and the lower-lying dashes to the asymptotic expressions. We also show for comparison the change in the corresponding coefficient for the separation K_1/K_1° (the two middle curves marked with circles). Here the quantities $K_1^\circ = \sqrt{\pi\varepsilon\rho}\,p$, $K_2^\circ = \sqrt{\pi\varepsilon\rho}\,t_2$, $K_3^\circ = \sqrt{\pi\varepsilon\rho}\,t_1$ are the intensity factors of the plane limit problems. As is seen from Fig.3, the accuracy of the asymptotic formulas for $\varepsilon < 0.25$ is around 1%, and increases rapidly as ε decreases.

Table 1

ζ	ε	K_2/K_2°	K_3/K_3°
0.5	$1/3$	0.9559 (0.9786)	0.8853 (0.8719)
	$1/4$	0.9729 (0.9092)	0.9280 (0.8959)
1.5	$1/3$	0.9092 (0.9407)	0.7947 (0.8811)
	$1/4$	0.9422 (0.9168)	0.8621 (0.8676)
2	$1/3$	0.8897 (0.9298)	0.7621 (0.7814)
	$1/4$	0.9290 (0.8946)	0.8362 (0.8641)

The table shows the coefficients K_2/K_2° and K_3/K_3°, obtained by using a numerical solution* (*The numerical solutions of the crack problems used for the comparison are constructed in: Brutyan, A.R., Gol'dshtein, R.V. and Fedorenko R.P., Statics and kinetics of spatial shear cracks. Preprint No.88, Inst. Prikl. Matem. Akad. Nauk SSSR, Moscow, 1985.) (indicated in the parentheses) and the asymptotic values in the section $x_1 = 0$ for cracks in the shape of a generalized ellipse /3/. As is seen from the table, the asymptotic values of the quantities presented increase as the parameter ζ decreases. This is associated with the fact that the domains of cracks with large ζ are inscribed in the domains of cracks with smaller ζ (see Sect.4).

2°. We consider an arbitrary uniformly loaded crack extended along a curve. In this case

$$Q_i(l, M, \varepsilon) = t_i(l)(1 - 0.25\alpha_j^{\,3}\lambda_j^{-1}\varepsilon\chi M) + \varepsilon^2 [Q_{ik}^{**} t_k(l)][16(1 + 6\delta_{1i})(1-\nu)]^{-1} \tag{3.6}$$

The expression for Q_{ik}^{**} (because of its awkwardness) is presented only for an annular crack of radius $R = \rho(l) = (R_1 + R_2)/2$ (where R_1 and R_2 are the outer and inner radii of the ring):

$$\begin{aligned} Q_{11}^{**} &= 6(1 - 13\nu + 4\nu^2)\ln 16/\varepsilon + 2(1-\nu)(9 + 20\nu)M^2 - 9 - \\ &\quad 113\nu + 160\nu^2 \\ Q_{22}^{**} &= (1 + 11\nu - 8\nu^2)\ln 16/\varepsilon + (3 - 15\nu)M^2 - 1.5 - 22.5\nu + 28\nu^2 \\ Q_{12}^{**} &= Q_{21}^{**} = 0 \end{aligned} \tag{3.7}$$

Therefore, the longitudinal and transverse shear problems become distinct for an annular crack. It is also interesting to note that according to (3.6) and (2.10) K_2 can be larger (in absolute value) in maximal-width sections of a uniformly loaded curved crack at both the inner ($M = -1$) and outer ($M = 1$) contours (depending on the sign of the coefficient $1 - 3\nu$) while K_2 and K_1 are always greater on the inner contour for separation cracks /3/.

3°. We consider an annular crack on whose edge the load has the form $\mathbf{t} = A(R + \varepsilon m)\,\boldsymbol{\tau}(l)$, which corresponds to torsion around an axis parallel to Ox_3 and passing through the centre of the crack. In this case (2.8) and (2.10) yield

$$\begin{aligned} u_1(l, m, \varepsilon) &= AR\mu^{-1}\varepsilon\sqrt{\rho^2(l) - m^2}\,Q_1(M, \varepsilon) + o(\varepsilon^3) \\ K_3(l, \pm R, \varepsilon) &= \pm\sqrt{\pi\varepsilon\rho}\,AR\,[Q_1(\pm 1, \varepsilon) + o(\varepsilon^2)] \\ Q_1(M, \varepsilon) &= 1 + 0.25\varepsilon M - 3\varepsilon^2(M^2 - \ln 16/\varepsilon)\,16 + (21 - 19\nu)\,\varepsilon^2//[96(1-\nu)] \end{aligned}$$

$u_2 = 0$, $K_2 = 0$ because of the symmetry of the problem. As ν grows the quantities u_1 and K_3 grow insignificantly in absolute value.

4°. We will establish certain properties of the asymptotic solution obtained that are analogous to comparison theorems for separation cracks /2/. We will henceforth limit ourselves to the case of an identical uniform load for all the cracks.

Let the cracks $G_1(\varepsilon)$, $G_2(\varepsilon)$ be extended along the very same curve $\mathrm{R}(l)$ and $G_2(\varepsilon) \supset G_1(\varepsilon)$ (Fig.4). It can obviously be assumed that the function $\rho_1(l)$ is given on the whole $[-L, L]$ by predefining it by zero on $[-L, a) \cup (b, L]$ (see Fig.4). Then $\rho_1(l) \leqslant \rho_2(l)$, $\forall l \in [-L, L]$. Here if $G_1(\varepsilon) \neq G_2(\varepsilon)$ (i.e., $\rho_1 \not\equiv \rho_2$ on $[-L, L]$), we shall say that $G_1(\varepsilon)$ is inscribed in $G_2(\varepsilon)$. Meanwhile if $\rho_1(l_0) = \rho_2(l_0)$, $l_0 \in (a, b)$, then the contours of $G_1(\varepsilon)$ and $G_2(\varepsilon)$ are tangent in the section $l = l_0$ ($\rho_1'(l_0) = \rho_2'(l_0)$). Let us consider cracks $G_1(\varepsilon)$ and $G_2(\varepsilon)$ extended along a line, where $G_1(\varepsilon)$ is inscribed in $G_2(\varepsilon)$. If $\rho_1(x_1^\circ) = \rho_2(x_1^\circ)$, $x_1^\circ \in (a, b)$, then by comparing terms of the

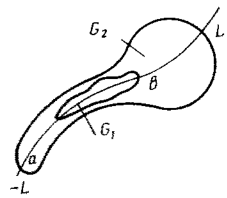

Fig.4

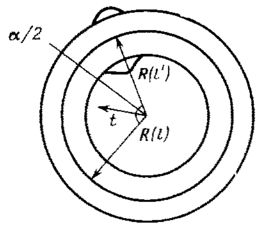

Fig.5

asymptotic form (3.3) for two cracks, it can be shown that for small ε the following inequality holds:

$$0 < c < Q_i^1(x_i^\circ, \varepsilon)/t_i < Q_i^2(x_1^\circ, \varepsilon)/t_i \quad (\nu \neq 0.5) \tag{3.8}$$

where the arbitrary constant is $c \in (0, 1)$. Hence, by taking account of (1.5) and (3.2), we have for sufficiently small ε and $x_1 \in (a, b)$, $|Y| < \min(\rho_1(x_1), \rho_2(x_1))$

$$\begin{gathered} |u_i^1(x_1, Y, \varepsilon)| < |u_i^2(x_1, Y, \varepsilon)| \\ |K_{i+1}^1(x_1, \pm\varepsilon\rho_1(x_1))| < |K_{i+1}^2(x_1, \pm\varepsilon\rho_2(x_1))| \\ (t_i \neq 0, \ \nu \neq 0.5) \end{gathered} \tag{3.9}$$

The inequalities (3.9) are analogues of comparison theorems for separation cracks.

We will now examine how the quantity

$$\delta W/\delta S = \beta w[(1-\nu)K_2^2 + K_3^2] \quad (\beta w = 0.5\mu^{-1}) \tag{3.10}$$

which is the specific increment of the total potential energy of a body during the advancement of a shear crack (as is known, its value governs the possibility of crack growth at a given point of the contour), will change as the crack domain varies. Taking account of (3.9) and (3.10), we have for cracks $G_1(\varepsilon)$ and $G_2(\varepsilon)$ extended along a line such that $G_1(\varepsilon)$ is inscribed in $G_2(\varepsilon)$, for $\forall x_1 \in (a, b)$ and sufficiently small ε

$$\begin{gathered} (\delta W_1/\delta S)(x_1, \pm\varepsilon\rho_1(x_1)) < (\delta W_2/\delta S)(x_1, \pm\varepsilon\rho_2(x_1)) \\ (\nu \neq 0.5, \ \mathbf{t} \neq 0) \end{gathered} \tag{3.11}$$

For the general case of cracks $G_1(\varepsilon)$, $G_2(\varepsilon)$ extended along a curve ($G_1(\varepsilon)$ is inscribed in $G_2(\varepsilon)$), the analogous inequality

$$(\delta W_1/\delta S)(l, \pm\rho_1(l), \varepsilon) < (\delta W_2/\delta S)(l, \pm\rho_2(l), \varepsilon) \quad (\mathbf{t} \neq 0) \tag{3.12}$$

is generally false, as will be seen later. However, we will show that (3.12) holds (for sufficiently small ε) for $\nu < \nu_0 = {}^1/_2(5 - \sqrt{17}) \approx 0.43845$. In the cases $\rho_1(l) < \rho_2(l)$ and $\rho_1(l) = \rho_2(l)$, $f_1''(l) < f''_2(l)$ the validity of the inequality (3.12) for sufficiently small ε (for $\nu \neq 0.5$) is established by using (2.9), (2.10), (3.10) by comparing terms of the asymptotic form $\delta W_1/\delta S$ and $\delta W_2/\delta S$ of orders ε and $\varepsilon^3 \ln \varepsilon$, respectively. In the case when $\rho_1(l) = \rho_2(l)$, $f_1''(l) = f_2''(l)$ the equation

$$\begin{gathered} (\delta W_2/\delta S)(l, \pm\rho_2(l), \varepsilon) - (\delta W_1/\delta S)(l, \pm\rho_1(l), \varepsilon) = 0.25\beta w\varepsilon^3 I^*(f_2 - f_1) + o(\varepsilon^3) \\ I^*(f_2 - f_1) = \int_{-L}^{L} B(l', l)|\Delta\mathbf{R}|^{-3}[f_2(l') - f_1(l')]\,dl' \\ B(l', l) = (1-2\nu)(\gamma(l'), \gamma(l)) + 3\nu(\gamma(l'), \mathbf{e}_{\Delta\mathbf{R}})(\gamma(l), \mathbf{e}_{\Delta\mathbf{R}}), \ \mathbf{e}_{\Delta\mathbf{R}} = \Delta\mathbf{R}/|\Delta\mathbf{R}| \end{gathered} \tag{3.13}$$

follows from (2.9), (2.10), (3.10).

We show that $B(l', l) > 0$ for $\nu < \nu_0$. We then obtain $I^*(f_2 - f_1) > 0$ and the inequality (3.12) will be proved for small ε.

Let φ be the angle between $\gamma(l')$ and $\gamma(l)$. Then

$$\varphi/2 \leqslant \arccos[2\sqrt{1-\nu}/(2-\nu)] < \pi/2 \tag{3.14}$$

Indeed, let ψ be the angle between $\mathbf{t}$ and $\gamma(l)$, where $\operatorname{tg}\psi = \nu \operatorname{tg}\alpha/(\operatorname{tg}^2\alpha + 1 - \nu)$, and α is the angle between $\mathbf{t}$ and $\mathbf{n}(l)$. Considering the extrema of tan ψ as a function of tan α, we obtain the inequality $\psi \leqslant \arccos[2\sqrt{1-\nu}/(2-\nu)]$. An analogous estimate holds for the angle between $\mathbf{t}$ and $\gamma(l')$. We hence obtain (3.14).

Let φ_1, φ_2 be the angles made by $\mathbf{e}_{\Delta\mathbf{R}}$ with $\gamma(l), \gamma(l')$, respectively, and $\varphi_2 - \varphi_1 = \varphi$. Then

$$\begin{gathered} B(l, l') = |\gamma(l')|\,|\gamma(l)|\,[(1-2\nu)\cos\varphi + 3\nu\cos\varphi_1\cos\varphi_2] = \\ 0.5|\gamma(l')|\,|\gamma(l)|\,[(2-\nu)\cos\varphi + 3\nu\cos(\varphi_1 + \varphi_2)] \end{gathered}$$

According to (3.14)

$$\cos\varphi = 2\cos^2(\varphi/2) - 1 \geqslant 8(1-\nu)(2-\nu)^{-2} - 1$$
$$(2-\nu)\cos\varphi + 3\nu\cos(\varphi_1+\varphi_2) \geqslant 8(1-\nu)/(2-\nu) - 2 + \nu - 3\nu =$$
$$2(\nu^2 - 5\nu + 2)/(2-\nu) > 0 \quad \text{for} \quad \nu < \nu_0$$

The proof of inequality (3.12) is completed.

By starting from the above exposition, an example can be constructed for $\forall\nu > \nu_0$ in which inequality (3.12) does not hold. Thus, an annular crack can be taken as $G_1(\varepsilon)$ and an annular crack with a tiny "expansion" (Fig.5) as $G_2(\varepsilon)$. Selection of the location of the point $\mathbf{R}(l)$ (in which (3.12) is not satisfied), and the "expansion", as well as the direction of the load $\mathbf{t}$ is shown in Fig.5, where the relationship $\sigma/2 = \operatorname{arc\,tg}\sqrt{1-\nu}$ should be satisfied.

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FINITE-PART INTEGRALS IN PROBLEMS OF THREE-DIMENSIONAL CRACKS*

A.M. LIN'KOV and S.G. MOGILEVSKAYA

An effective method is proposed for solving the boundary integral equation (BIE) for the problem of a crack along a curvilinear surface in an elastic space on the basis of the transformation of the initial integrodifferential equation into an equation without derivatives. This is achieved by using the concept of the finite-part integral (FPI). Quadrature formulas are presented for such integrals over arbitrary convex polygons by approximating displacement discontinuities on the boundary by polynomials.

The well-known BIE for three-dimensional cracks contain either derivatives of the unknown functions or derivatives of a surface integral /1-7/. In both cases the presence of the derivatives significantly complicates the solution. However, as is shown in /8/, these difficulties are reduced in the case of a plane crack of normal discontinuity if the FPI concept is utilized /9, 10/. In this connection, it is useful to investigate the possibility of applying such an approach to the more general problem of a crack of arbitrary discontinuity and to develop the numerical side of its utilization. Both aims are pursued in this paper: the extension of this idea to the general case of three-dimensional cracks is given and methods are indicated for evaluating the integrals that originate by presenting quadrature formulas convenient for the numerical realization of the BIE method on a computer.

1. The consideration of the problem is based on the form of the BIE for three-dimensional cracks, which contains only derivatives of integrals over the surface but no derivatives of the displacement discontinuities under the integral sign /1, 6/. The integrals in the BIE have singularities generated by the term $1/r$ and combinations of its powers with differences between the coordinates of the control point x and the variable point of integration ξ (r is the distance between the points). This does not permit differentiation under the integral sign since it results in a non-integrable singularity (in the general case the original

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